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# WAVE EVOLUTION, REFLECTION, AND TRANSMISSION ALONG INHOMOGENEOUS WAVEGUIDES

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One-dimensional wave propagation along an inhomogeneous waveguide, such as an acoustic horn, a plate strip, a rod, or a beam is considered. The equations of motion of a generic waveguide are written in first order form, and the properties of the (position-dependent) eigenvalues and eigenvectors of the system are deduced by considering the symmetry and/or the energetics of the system. The equations of motion are then transformed to "wave co-ordinates" based on the right eigenvectors of the system, and a perturbation method is employed to study wave propagation along deterministic, periodic, and random waveguides. Attention is then turned to wave reflection at a "cut-off" cross-section, and a general result for the phase of the reflection coefficient is derived. The general theory is illustrated by application to rod and beam examples. The method can be likened to a "coupled modes" approach in which the properties of the eigenvectors (or "modes") arising from symmetry and/or energetics are exploited. © 1999 Academic Press

## 1. INTRODUCTION

This work is concerned with wave propagation along an inhomogeneous one-dimensional waveguide, such as an acoustic horn, a plate strip, a rod, or a beam. This topic is certainly not a new one, and previous studies have considered wave evolution, reflection, and transmission, in various specific waveguides, together with the related topic of the modes of vibration of a finite system. As described in what follows, the aim of the present work is to provide a straightforward methodology for this type of analysis by employing a general "coupled modes" approach in conjunction with a study of the system energetics.

Much previous work on wave propagation in an inhomogeneous waveguide has been related to acoustic horns or ducts. For example, Lighthill [1] has presented a study of wave propagation along a slowly varying acoustic duct: it is shown that to a first approximation the wave amplitude evolves to preserve energy flow, and there is no wave reflection. For more rapidly varying ducts significant wave reflection can occur, and Pierce [2] has summarized earlier work by Karal [3] and Miles [4, 5] regarding wave transmission across a fairly rapid change in the duct cross-sectional area. Whereas Lighthill [1] considered a one-dimensional longitudinal wave model of the duct, references [4, 5]

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present a detailed two-dimensional analysis, and the same two-dimensional problem is addressed in reference [6] by using the method of matched asymptotic expansions. The longitudinal wave model of the duct [1] has the same mathematical form as the equation of a non-uniform rod, and a number of recent papers [7–10] have considered waves and modes in such systems. The same form of equation was employed by Scott [11] to study the statistics of wave propagation in a one-dimensional random medium. References [9, 10] consider the further case of a non-uniform beam, although in this case the focus is on the vibration modes rather than wave propagation. In references [1, 7–11] the solution procedure is based directly on the governing one-dimensional second-order differential equation of a duct, rod, or waveguide, or the one-dimensional fourth-order differential equation of a beam. In contrast, Galanenko [12] has considered wave propagation in an elastic waveguide by employing a "coupled modes" approach, in which the governing equations are expressed initially in first-order form. Galanenko suggests that although this type of approach has previously been employed in underwater acoustics (for example reference [13]), reference [12] represents the first application of the method to structural systems. The method allows very general waveguides to be considered: the cross-sectional motion is expressed in terms of generalized co-ordinates, which may correspond to physical quantities (as in the case of a rod or a beam) or to cross-sectional Ritz functions. With regard to an acoustic duct, the method could therefore encompass either a simple longitudinal wave model, or a more detailed two-dimensional acoustic analysis, as adopted in references [3–6].

In the present work, the equations of motion of a waveguide are expressed in first-order form and then transformed to "wave co-ordinates" by considering the (position-dependent) eigenvectors and eigenvalues of the system. Various properties of the eigenvectors and eigenvalues, and hence of the transformed equations of motion, are deduced by considering the symmetry and/or the energetics of the system. These properties are then exploited in considering a perturbation solution of the equations of motion for various cases, which include a slowly varying waveguide, a periodic waveguide, and a random waveguide. Special attention is given to the case of wave reflection at a "cut-off" cross-section, and it is shown that a result obtained by Scott and Woodhouse [14] for the special case of internal reflection in a doubly curved plate strip is in fact a general result. The method of obtaining the transformed equations of motion owes much to that proposed by Galanenko, although energetics were not considered in detail in reference [12]. The results obtained are illustrated by application to a rod and a beam.

The equations of motion of a waveguide are developed in section 2, which includes a study of the effects of symmetry and energy conservation on the structure

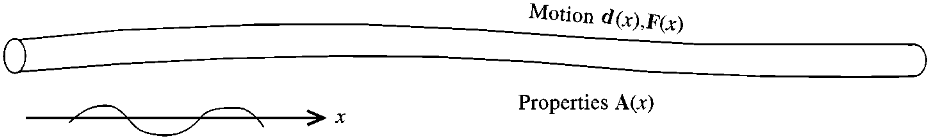



Figure 1. Schematic of an inhomogeneous waveguide.

of the equations. A perturbation solution to the equations is then presented in section 3, and this is applied to wave evolution, reflection, and transmission in deterministic, periodic, and random waveguides. Wave reflection at a cut-off cross-section is then considered in section 4.

## 2. WAVEGUIDE EQUATIONS OF MOTION

### 2.1. GENERAL FORM OF THE EQUATIONS

The present work is concerned with the dynamics of a one-dimensional waveguide such as a beam, rod, or acoustic tube. A schematic of the waveguide is shown in Figure 1, where the $N \times 1$ vector $\mathbf{d}(x)$ represents the set of displacements which are required to describe the motion at location $x$ and the $N \times 1$ vector $\mathbf{F}(x)$ represents the associated set of elastic forces. The motion of the waveguide is taken to be simple harmonic with frequency $\omega$ so that all variables are assumed to vary with time in proportion to the factor $\exp(\mathrm{i}\omega t)$; this implies that the entries of $\mathbf{d}$ and $\mathbf{F}$ represent complex amplitudes which contain information concerning both the phase and the magnitude of the motion. The displacements $\mathbf{d}(x)$ may represent either the physical displacements of the waveguide cross-section, as in the case of a simple rod or beam, or generalized co-ordinates for the case of more complex systems in which the cross-section motion is modelled by using admissible functions.

Regardless of the physical nature of the waveguide, the governing equations of motion can always be written in terms of the $2N \times 1$ state vector $\mathbf{u}(x) = (\mathbf{d}^{\mathrm{T}}\ \mathbf{F}^{\mathrm{T}})^{\mathrm{T}}$ in the canonical form

$$\mathrm{d}\mathbf{u}/\mathrm{d}x = \mathbf{A}\mathbf{u}, \tag{1}$$

where $\mathbf{A}(x)$ is a $2N \times 2N$ matrix. The aim of the present paper is to consider the solution of equation (1) for a range of situations.

Throughout the present work, two specific examples will be employed to illustrate the general results which are obtained. The first of these example concerns the axial motion of an elastic rod, for which the governing differential equation has the well-known form [15]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ES\frac{\mathrm{d}w}{\mathrm{d}x}\right) + \rho S\omega^2 w = 0, \tag{2}$$

where $w$ is the axial displacement and $E$, $S$, and $\rho$ are respectively the Young's modulus, the cross-sectional area, and the density. Equation (2) can be written in the form of equation (1) by defining

$$\mathbf{u} = \begin{pmatrix} w \\ ES(\mathrm{d}w/\mathrm{d}x) \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 0 & 1/(ES) \\ -\rho S\omega^2 & 0 \end{pmatrix}. \tag{3}$$

By an appropriate change in notation, equations (2) and (3) can also be taken to represent a tensioned string or the propagation of the lowest cross-sectional mode of an acoustic tube. In the case of an acoustic tube, the form of the matrix **A** explains the phenomenon of the "dual horn": as noted by Ram and Elhay [10]; Benade [16] stated "consider a pair of horns such that the product $A_1A_2$ of their cross-sectional areas is a constant from one end to the other...If the small ends of both of these horns are closed off, the two air columns turn out to have identical natural frequencies". Replacing $S$ by $1/S$ in equation (3) results in an identical acoustic system providing the pressure and displacement variables (the analogies of $w$ and $ES\mathrm{d}w/\mathrm{d}s$) are interchanged: thus a horn with a cross-sectional variation $S(x)$ and a blocked left-hand end will have the same natural frequencies as a horn with a cross-sectional variation $1/S(x)$ and a blocked right-hand end.

The second specific example concerns a pre-compressed beam which rests on an elastic foundation. In this case the equation of motion has the form [15]

$$\frac{\mathrm{d}}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right)+\frac{\mathrm{d}}{\mathrm{d}x}\left(C\frac{\mathrm{d}w}{\mathrm{d}x}\right)-(\rho S\omega^2-K)w=0, \tag{4}$$

where $w$ is the lateral deflection, and in addition to the symbols previously defined, $I$ is the second moment of area, $C$ is the pre-compression, and $K$ is the foundation stiffness per unit length. Equation (4) can be cast into the form of equation (1) by employing the following notation:

$$\mathbf{u}=\begin{pmatrix} w \\ \mathrm{d}w/\mathrm{d}x \\ -(\mathrm{d}/\mathrm{d}x)[EI(\mathrm{d}^2w/\mathrm{d}x^2)]-C(\mathrm{d}w/\mathrm{d}x) \\ EI(\mathrm{d}^2w/\mathrm{d}x^2) \end{pmatrix},$$

$$\mathbf{A}=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/(EI) \\ -\rho S\omega^2+K & 0 & 0 & 0 \\ 0 & -C & -1 & 0 \end{pmatrix}. \tag{5,6}$$

As a preliminary to considering the solution of equation (1), the properties of the matrix **A** are considered in sections 2.2–2.4.

### 2.2. SYMMETRIC SYSTEMS

A finite section of the waveguide covering the region $x$ to $x+X$ is shown in Figure 2. The terms $(\mathbf{d}_L\ \mathbf{F}_L)$ and $(\mathbf{d}_R\ \mathbf{F}_R)$ which appear in this figure represent the displacements and elastic forces at the left- and right-hand sides of the section respectively. The dynamic stiffness matrix $\mathbf{D}(x, X)$ and the transfer matrix $\mathbf{T}(x, X)$

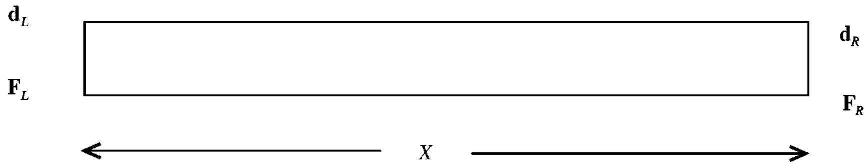



Figure 2. A finite section of the waveguide.

of the section are defined by the equations

$$\begin{pmatrix} -\mathbf{F}_L \\ \mathbf{F}_R \end{pmatrix} = \mathbf{D}\begin{pmatrix} \mathbf{d}_L \\ \mathbf{d}_R \end{pmatrix}, \quad \begin{pmatrix} \mathbf{d}_R \\ \mathbf{F}_R \end{pmatrix} = \mathbf{T}\begin{pmatrix} \mathbf{d}_L \\ \mathbf{F}_L \end{pmatrix} \tag{7,8}$$

It follows that **D** can be expressed in terms of **T**, and *vice versa* [17]. The vast majority of physical systems obey the principle of reciprocity, which implies that the dynamic stiffness matrix **D** is symmetric; in this case it can readily be shown that the transfer matrix **T** is symplectic, which means that it has the property [17, 18]

$$\mathbf{T}^{\mathrm{T}}\mathbf{J}\mathbf{T} = \mathbf{J}, \quad \mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}, \tag{9,10}$$

where **I** is the $N \times N$ identity matrix. Now by noting that $\mathbf{T}(x, 0) = \mathbf{I}$, where in this case **I** represents the $2N \times 2N$ identity matrix, it follows from equations (1) and (8) that

$$\mathbf{A}(x) = \frac{\mathrm{d}}{\mathrm{d}X}\mathbf{T}(x, X)|_{X=0}. \tag{11}$$

By differentiating equation (9) with respect to $X$ and then placing $X = 0$, it then follows that a symmetric system (i.e. a system with a symmetric dynamic stiffness matrix **D**) has the property

$$\mathbf{J}\mathbf{A} = -\mathbf{A}^{\mathrm{T}}\mathbf{J}, \tag{12}$$

which implies that the matrix **JA** is symmetric. As will be discussed in section 2.4, this property has significant implications regarding the structure of the eigenvalues and eigenvectors of **A**, and this in turn reflects upon the physical nature of wave motion in the system. In particular, it is found that waves occur in left- and right-going pairs, which is consistent with the principle of reciprocity and the symmetry of **D**.

### 2.3. ENERGY FLOW, ENERGY CONSERVATION AND ENERGY DENSITY

The time-average energy flow $P(x)$ through the waveguide at location $x$ can be written as $P(x) = \langle \text{velocity} \times \text{force} \rangle$. The force vector has complex amplitude **F**, and given that the motion is harmonic with frequency $\omega$, the velocity vector has complex amplitude i$\omega$**d**. The time-averaged energy flow thus has the form

$$P(x) = (1/2)\mathrm{Re}\{\mathrm{i}\omega\, \mathbf{d}^{*\mathrm{T}}\mathbf{F}\} = (1/4)\mathrm{i}\omega\, \mathbf{u}^{*\mathrm{T}}\mathbf{J}\mathbf{u}, \tag{13}$$

where **J** is given by equation (10). If the system is conservative then the energy flow must be constant, so that

$$\frac{\mathrm{d}}{\mathrm{d}x}P(x) = (1/4)\mathrm{i}\omega\left(\frac{\mathrm{d}}{\mathrm{d}x}(\mathbf{u}^{*\mathrm{T}})\mathbf{J}\mathbf{u} + \mathbf{u}^{*\mathrm{T}}\mathbf{J}\frac{\mathrm{d}}{\mathrm{d}x}(\mathbf{u})\right) = (1/4)\mathrm{i}\omega\, \mathbf{u}^{*\mathrm{T}}[\mathbf{A}^{*\mathrm{T}}\mathbf{J} + \mathbf{J}\mathbf{A}]\mathbf{u} = 0. \tag{14}$$

Equation (14) must hold for any value of the state vector **u**, and hence it can be deduced that a conservative system must have the property

$$\mathbf{J}\mathbf{A} = -\,\mathbf{A}^{*\mathrm{T}}\mathbf{J}. \tag{15}$$

In this case the matrix **JA** is Hermitian rather than symmetric, and it can be shown that the transfer matrix **T** of any section of the waveguide is K-unitary rather than symplectic [17, 18], so that $\mathbf{T}^{*\mathrm{T}}\mathbf{J}\mathbf{T} = \mathbf{J}$. The effect of this property on the eigenvalues and eigenvectors of **A** is discussed in section 2.4. As an aside, it can be noted from equations (12) and (15) that **A** must be real if the system is to be both symmetric and conservative.

By considering the energy flow in a lightly damped system it is possible to derive an expression for the kinetic energy density of the waveguide. A small amount of mass proportional damping can be modelled by considering the vibration frequency $\omega$ to be replaced by the complex quantity $\omega(1 - \mathrm{i}\eta/2)$, where $\eta$ is the loss factor. The matrix **A** then becomes $\mathbf{A}_0 - (\mathrm{i}\eta\omega/2)\partial\mathbf{A}_0/\partial\omega$, where $\mathbf{A}_0$ is the matrix associated with the undamped system. By noting from equation (15) that $\mathbf{J}(\partial\mathbf{A}_0/\partial\omega)$ is a Hermitian matrix, equation (14) can in this case be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}P(x) = (1/4)\eta\omega^2\mathbf{u}^{*\mathrm{T}}\mathbf{J}(\partial\mathbf{A}/\partial\omega)\mathbf{u}, \tag{16}$$

where the subscript on $\mathbf{A}_0$ has been dropped for ease of notation. Now for mass proportional damping it is known that the power dissipated by a small element d$x$ of the waveguide can be written as $2\omega\eta\bar{T}\mathrm{d}x$ where $\bar{T}$ is the time-averaged kinetic energy density; the power dissipated can also be written as $-(\mathrm{d}P/\mathrm{d}x)\mathrm{d}x$, and thus it follows from equation (16) that the kinetic energy density is given by

$$\bar{T} = -(1/8)\omega\mathbf{u}^{*\mathrm{T}}\mathbf{J}(\partial\mathbf{A}/\partial\omega)\mathbf{u}. \tag{17}$$

This expression can be compared with a result derived in reference [17] for the kinetic energy stored in a finite section of a waveguide. Equation (5.8) of reference [17] expresses the kinetic energy in terms of the transfer matrix **T**: if the length of the section, d$x$ say, is allowed to tend to zero so that $\mathbf{T} \approx \mathbf{I} + \mathbf{A}\mathrm{d}x$, then equation (17) of the present work is recovered. Furthermore, it can readily be confirmed that for each of the example systems presented in section 2.1, equation (17) yields the anticipated result $\bar{T} = (1/4)\omega^2 \rho S|w|^2$.

### 2.4. EIGENVALUES AND EIGENVECTORS OF A

If the system is *symmetric*, so that **A** satisfies equation (12), then it follows that

$$\mathbf{A}\mathbf{u}_j = \lambda_j \mathbf{u}_j \;\Rightarrow\; (\mathbf{J}\mathbf{u}_j)^{\mathrm{T}}\mathbf{A} = -\lambda_j (\mathbf{J}\mathbf{u}_j)^{\mathrm{T}}. \tag{18}$$

This means that any eigenvalue $\lambda_j$ of **A** with right eigenvector $\mathbf{u}_j$ is accompanied by an eigenvalue $-\lambda_j$ with left eigenvector $\mathbf{J}\mathbf{u}_j$. Physically, the eigenvalues of **A** are the wavenumbers associated with wave motion in a homogeneous waveguide: a purely imaginary value of $\lambda_j$ represents a propagating wave, while a non-zero real part will lead to either exponential growth or exponential decay. Equation (18) implies that the waves occur in left- and right-going pairs, which is, of course, fully consistent with the symmetry of the system.

If the system is *conservative*, so that equation (15) is applicable, then it follows that

$$\mathbf{A}\mathbf{u}_j = \lambda_j \mathbf{u}_j \;\Rightarrow\; (\mathbf{J}\mathbf{u}_j^*)^{\mathrm{T}}\mathbf{A} = -\lambda_j^* (\mathbf{J}\mathbf{u}_j^*)^{\mathrm{T}}. \tag{19}$$

In this case the eigenvalues occur in paris of the form $(\lambda_j, \; -\lambda_j^*)$; the right eigenvector associated with $\lambda_j$ is $\mathbf{u}_j$ and the left eigenvector associated with $-\lambda_j^*$ is $\mathbf{J}\mathbf{u}_j^*$. It can be noted that if $\lambda_j$ is purely imaginary then $\lambda_j = -\lambda_j^*$ and the "pair" of waves reduces to a single propagating wave. Certain aspects of the energy flow in a conservative system can investigated by expanding the state vector **u** in terms of the right eigenvectors $\mathbf{u}_j$, so that

$$\mathbf{u} = \sum_j b_j \mathbf{u}_j, \tag{20}$$

where $b_j$ are the appropriate expansion coefficients. The energy flow follows from equation (13) in the form

$$P = \sum_j \sum_r b_j^* b_r P_{jr}, \quad P_{jr} = (1/4)\mathrm{i}\omega \mathbf{u}_j^{*\mathrm{T}} \mathbf{J}\mathbf{u}_r. \tag{21, 22}$$

Now it can readily be shown that $P_{jr} = 0$ unless $\lambda_r = -\lambda_j^*$, which means that energy flow can arise only as a result of interaction between two members of an eigenvalue pair $(\lambda_j, \; -\lambda_j^*)$. In the special case of a propagating wave with $\lambda_j$ purely imaginary, the "pair" reduces to a single wave with $P_{jj} \neq 0$, so that the energy flow is of course non-zero. Equivalent results regarding energy flow interaction have been derived previously in references [17, 19] via a transfer matrix analysis.

As a final comment in this section, it can be noted that if the system is symmetric *and* conservative, then both equations (18) and (19) apply, and eigenvalue groups of the type $(\pm\lambda_j \; \pm\lambda_j^*)$ arise. If $\lambda_j$ is either purely real or purely imaginary then the group of four eigenvalues reduces to just two distinct values $(\lambda_j, \; -\lambda_j)$.

2.5. WAVE AMPLITUDE EQUATIONS OF MOTION

Were the matrix $\mathbf{A}$ to commute with its integral over $x$, then equation (1) could be solved very simply by employing a matrix exponential integrating factor. In general, $\mathbf{A}$ does not have this property, and a simple closed-form solution to equation (1) cannot be found. A form of the equation which is more amenable to the development of approximate analytical solutions can be derived by applying the change of variables represented by equation (20), so that

$$\mathbf{u} = \mathbf{\Phi}\mathbf{b}, \tag{23}$$

where the columns of the matrix $\mathbf{\Phi}$ contain the right eigenvectors of $\mathbf{A}$, and $\mathbf{b}$ is a new state vector which contains the amplitudes of the eigenvectors. Given that the eigenvectors provide a description of wave motion in a homogeneous waveguide, the entries of $\mathbf{b}$ can be conveniently referred to as (complex) wave amplitudes. By substituting equation (23) into equation (1), it can be shown that $\mathbf{b}$ must satisfy the differential equation

$$\mathrm{d}\mathbf{b}/\mathrm{d}x = \mathbf{\Lambda}\mathbf{b} - \mathbf{\Phi}^{-1}(\mathrm{d}\mathbf{\Phi}/\mathrm{d}x)\mathbf{b}, \tag{24}$$

where $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\mathbf{A}$. Now the second term on the right of this equation can be re-expressed by noting that

$$\mathbf{A\Phi} = \mathbf{\Phi\Lambda} \;\Rightarrow \mathbf{\Lambda\Phi}^{-1}(\mathrm{d}\mathbf{\Phi}/\mathrm{d}x) - \mathbf{\Phi}^{-1}(\mathrm{d}\mathbf{\Phi}/\mathrm{d}x)\mathbf{\Lambda} = (\mathrm{d}\mathbf{\Lambda}/\mathrm{d}x) - \mathbf{\Phi}^{-1}(\mathrm{d}\mathbf{A}/\mathrm{d}x)\mathbf{\Phi}. \tag{25}$$

This yields

$$[\mathbf{\Phi}^{-1}(\mathrm{d}\mathbf{\Phi}/\mathrm{d}x)]_{jr} = [\mathbf{\Phi}^{-1}(\mathrm{d}\mathbf{A}/\mathrm{d}x)\mathbf{\Phi}]_{jr}/(\lambda_r - \lambda_j), \quad j \neq r, \tag{26}$$

where the notation $[\mathbf{X}]_{jr}$ indicates the $jr$th entry of the matrix $\mathbf{X}$. The off-diagonal contributions to the right hand side of equation (24) can thus be expressed very simply in terms of the rate of change of the matrix $\mathbf{A}$. The diagonal components are less straight forward, but progress can be made by noting from section 2.3 that for a conservative system the *left* eigenvector associated with $\lambda_j$ is $\mathbf{J}\mathbf{u}_{j'}^*$, where $j'$ is such that $\lambda_{j'} = -\lambda_j^*$. This implies that the $j$th row of $\mathbf{\Phi}^{-1}$ has the form $\mathbf{u}_{j'}^{*\mathrm{T}}\mathbf{J}/(\mathbf{u}_{j'}^{*\mathrm{T}}\mathbf{J}\mathbf{u}_j)$ so that

$$[\mathbf{\Phi}^{-1}(\mathrm{d}\mathbf{\Phi}/\mathrm{d}x)]_{jj} = \mathbf{u}_{j'}^{*\mathrm{T}}\mathbf{J}(\mathrm{d}\mathbf{u}_j/\mathrm{d}x)/(\mathbf{u}_{j'}^{*\mathrm{T}}\mathbf{J}\mathbf{u}_j). \tag{27}$$

Now at this stage there is a degree of ambiguity regarding the definition of the wave amplitudes $\mathbf{b}$, since no scaling convention for the mode shapes $\mathbf{u}_j$ has as yet been defined. The form of equation (27) suggests that a scaling of the type $\mathbf{u}_{j'}^{*\mathrm{T}}\mathbf{J}\mathbf{u}_j =$ constant would be appropriate, and in fact it follows from equation (22) that the choice

$$\mathbf{u}_{j'}^{*\mathrm{T}}\mathbf{J}\mathbf{u}_j = -4\mathrm{i}\gamma_j/\omega \;\Rightarrow\; [\mathbf{\Phi}^{-1}(\mathrm{d}\mathbf{\Phi}/\mathrm{d}x)]_{jj} = (\mathrm{i}\omega\gamma_j/4)\mathbf{u}_{j'}^{*\mathrm{T}}\mathbf{J}(\mathrm{d}\mathbf{u}_j/\mathrm{d}x), \tag{28}$$

where $\gamma_j = \pm 1$ is particularly convenient, since this gives $P_{jr} = \gamma_j$ for all non-zero energy flows. The term $\gamma_j$ must be included in this result because it is possible to apply equation (28) only for *either* $\gamma_j = 1$ *or* $\gamma_j = -1$: for example, a wave which propagates energy to the right cannot be scaled to give $\gamma_j = -1$.

Equations (24), (26) and (28) together constitute the equations of motion for the wave amplitudes $\mathbf{b}$. It can be noted that the motion of a physical system will be determined by the solution of equation (24) under the appropriate boundary conditions. For the free vibrations of a finite system, for example, the solution of equation (24) subject to the boundary conditions at either end of the system will yield the natural frequencies and normal modes. The form of the eigenvectors $\mathbf{u}_j$ in equation (20) (or, physically, the wave motion in the system) is independent of any boundary conditions, but of course the way in which these eigenvectors combine to produce the physical response is strongly determined by the boundary conditions.

The approximate solution of equation (24) for the case of slowly varying waveguide is considered in the following section. The case of a rapidly varying waveguide is considered in section 4, together with the issue of wave reflection at a cut-off cross-section.

## 3. WAVE PROPAGATION IN A SLOWLY VARYING WAVEGUIDE

### 3.1. PERTURBATION SOLUTION OF THE WAVE AMPLITUDE EQUATIONS

The properties of the waveguide can be considered to be slowly varying if the second term of the right-hand side of equation (24) is relatively small. In more detail, if $\mathrm{O}(\mathrm{d}\boldsymbol{\Phi}/\mathrm{d}x) = \alpha\mathrm{O}(\boldsymbol{\Phi})$ then the term can be considered small if $\mathrm{O}(\alpha/\lambda_j) \ll 1$, which implies that the lengthscale of the waveguide variation is large in comparison to a wavelength. Under this condition, equation (24) can be rewritten in the form

$$\mathrm{d}\mathbf{b}/\mathrm{d}x = \boldsymbol{\Lambda}\mathbf{b} + \varepsilon\mathbf{B}\mathbf{b}, \quad \varepsilon\mathbf{B} = -\boldsymbol{\Phi}^{-1}(\mathrm{d}\boldsymbol{\Phi}/\mathrm{d}x), \tag{29, 30}$$

where $\mathrm{O}(\mathbf{B}) = \mathrm{O}(\boldsymbol{\Lambda})$ and $\varepsilon$ is a small parameter. A solution to equation (29) can be sought by expanding $\mathbf{b}$ in the form of a perturbation series so that

$$\mathbf{b} = \mathbf{b}_0 + \varepsilon\mathbf{b}_1 + \varepsilon^2\mathbf{b}_2 + \cdots. \tag{31}$$

If equation (31) is substituted into equation (29) and like powers of $\varepsilon$ are equated then the following hierarchy of equations is obtained:

$$\mathrm{d}\mathbf{b}_0/\mathrm{d}x = \boldsymbol{\Lambda}\mathbf{b}_0, \quad \mathrm{d}\mathbf{b}_n/\mathrm{d}x = \boldsymbol{\Lambda}\mathbf{b}_n + \mathbf{B}\mathbf{b}_{n-1}, \quad n > 0. \tag{32, 33}$$

This yields

$$\mathbf{b}_0 = \boldsymbol{\Gamma}(x)\mathbf{c}, \quad \mathbf{b}_n = \boldsymbol{\Gamma}(x)\int_0^x \boldsymbol{\Gamma}^{-1}(x')\mathbf{B}(x')\mathbf{b}_{n-1}(x')\,\mathrm{d}x' \quad n > 0, \tag{34, 35}$$

where the constant vector $\mathbf{c}$ is determined by the spatial boundary conditions, and

$$\boldsymbol{\Gamma}(x)=\exp\left(\int_0^x \boldsymbol{\Lambda}(x')\,\mathrm{d}x'\right). \tag{36}$$

Equations (34–36) are applied to a number of example cases in the following sections.

### 3.2. THE EVOLUTION OF A PROPAGATING WAVE

If the eigenvalue $\lambda_j$ is purely imaginary with $\lambda_j=-\mathrm{i}k_j$, then the $j$th wave component represents a right-propagating wave with wavenumber $k_j$. The evolution of this wave along a slowly varying waveguide can be investigated by applying equations (34–36) with $c_j=1$ and $c_r=0$ for $r\neq j$; this implies that only the $j$th wave component is present at $x=0$, and the wave has an initial complex amplitude of unity. In applying equations (34) and (35) to this case, it can be noted initially that the integrand of equation (35) will always contain a term of the form $\boldsymbol{\Gamma}^{-1}(x')\mathbf{B}(x')\boldsymbol{\Gamma}(x')$. The $jr$th component of this matrix can be written as

$$[\boldsymbol{\Gamma}^{-1}(x')\mathbf{B}(x')\boldsymbol{\Gamma}(x')]_{jr}=[\mathbf{B}(x')]_{jr}\exp\left\{\int_0^{x'}[\mathrm{i}k_j(x'')+\lambda_r(x'')]\,\mathrm{d}x''\right\}. \tag{37}$$

For $j\neq r$ the exponential term in this expression will oscillate in sign as $x'$ is varied; by definition the term $\mathbf{B}(x')$ will change at a relatively slow rate ($\mathrm{O}(\alpha/\lambda_j)\ll 1$ in the notation of the previous section), and this implies that the off-diagonal terms in equation (35) will integrate to produce a near zero result. If only the diagonal entries are retained in equation (37), i.e. $j=r$ so that $\lambda_r=-\mathrm{i}k_j$, then equations (31)–(36) yield

$$b_j(x)=\Gamma_j(x)\exp\left\{-\int_0^x[\boldsymbol{\Phi}^{-1}(\mathrm{d}\boldsymbol{\Phi}/\mathrm{d}x')]_{jj}\mathrm{d}x'\right\},\quad b_r(x)=0\quad r\neq j. \tag{38, 39}$$

This result could also have been derived directly from equation (24) by neglecting the coupling between the wave amplitudes $\mathbf{b}$; the justification for this approximation would of course require a similar argument to the one presented here. Now by noting that $j'\equiv j$ for a propagating wave, it follows from equation (28) that

$$[\boldsymbol{\Phi}^{-1}(\mathrm{d}\boldsymbol{\Phi}/\mathrm{d}x)]_{jj}=(\mathrm{i}\omega\gamma_j/4)\mathbf{u}_j^{*\mathrm{T}}\mathbf{J}(\mathrm{d}\mathbf{u}_j/\mathrm{d}x)=(\mathrm{i}\omega\gamma_j/4)\mathbf{u}_j^{\mathrm{T}}\mathbf{J}(\mathrm{d}\mathbf{u}_j^*/\mathrm{d}x). \tag{40}$$

This implies that the exponent in equation (38) is purely imaginary, and hence the modulus of $b_j$ is constant, although the phase may vary to a greater extent than that predicted by the simple "wavenumber" term $\boldsymbol{\Gamma}_j(x)$. In view of equations (21), (22), and (28), this result amounts to conservation of energy flow in the propagating wave, with no transfer of energy to any other wave component; this result is in agreement with the principles of ray acoustics [2] and is well known for simple waveguides [1]. The case in which the off-diagonal components of equation (37) are non-negligible, so that energy is transferred to other wave components, is considered in the following section.

A straightforward example of the present analysis is afforded by considering an elastic rod (or equivalently an acoustic tube) of varying cross-section $S$. In this case the matrix $\mathbf{A}$ is given by equation (3), and the eigenvalues and normalized eigenvectors have the form

$$\mathbf{\Lambda} = \begin{pmatrix} \mathrm{i}k & 0 \\ 0 & -\mathrm{i}k \end{pmatrix}, \quad \mathbf{\Phi} = (2ES\omega k)^{-1/2} \begin{pmatrix} 1 & 1 \\ \mathrm{i}kES & -\mathrm{i}kES \end{pmatrix}, \tag{41, 42}$$

where $k = \omega\sqrt{\rho/E}$. It follows that

$$\mathbf{\Phi}^{-1}(\mathrm{d}\mathbf{\Phi}/\mathrm{d}x) = -[(\mathrm{d}S/\mathrm{d}x)/(2S)] \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{43}$$

In this case equation (38) yields $b_j = \exp(-\mathrm{i}k_j x)$ so there is no change in phase over and above that associated with the wavenumber $k_j$. Since the modulus of $b_j$ is conserved, it follows from equation (42) that the physical displacement $w$ of the system must vary in proportion to $[S(x)]^{-1/2}$, a result which is well known for an acoustic horn [1]. The accuracy of this approximate result can be investigated numerically by considering the case

$$S(x) = S_0 + (1/2)(S_1 - S_0)\mathrm{erfc}[-(x - x_0)/(\sqrt{2}\sigma)], \tag{44}$$

where erfc is the complementary error function. Here the cross-sectional area changes from very nearly $S_0$ to very nearly $S_1$ over the range $(x_0 - 3\sigma) < x < (x_0 + 3\sigma)$; the ratio $n_\lambda = 3k\sigma/\pi$ gives an indication of the number of wavelengths taken to effect the change, while $S_R = S_1/S_0$ gives a measure of the magnitude of the change.

A typical example of wave propagation through the system is shown in Figure 3, as obtained by direct numerical integration of equation (1). The initial conditions consist of a *left* travelling wave of unit amplitude ($|b_1| = 1$) at the point $x = 0$, and the waveguide parameters are $n_\lambda = 1$ and $S_R = 2$. Because of the selected initial condition, the physical situation is that a left travelling wave enters the system at $kx = 12\pi$ and is partially transmitted and partially reflected by the change in the waveguide cross-section (centred on $kx_0 = 6\pi$); the initial condition at $x = 0$ represents the transmitted wave. The results shown in the figure are the real part of the physical displacement of the system $w$, together with the function $[S(x)/S_0]^{-1/2}$—the foregoing theory implies that $w$ should scale in proportion to $[S(x)/S_0]^{-1/2}$. In terms of the wave amplitudes, it is found that $|b_1| = 1$ at $kx = 0$ (by virtue of the imposed initial condition) and that $|b_1| = 1{\cdot}0008$ at $kx = 12\pi$. Were the wave to propagate without any energy transfer to the right-going wave, as predicted by equation (38), then clearly $b_1$ would have unit amplitude at all points of the waveguide—the value of $|b_1|$ at $kx = 12\pi$ is a measure of the validity of the

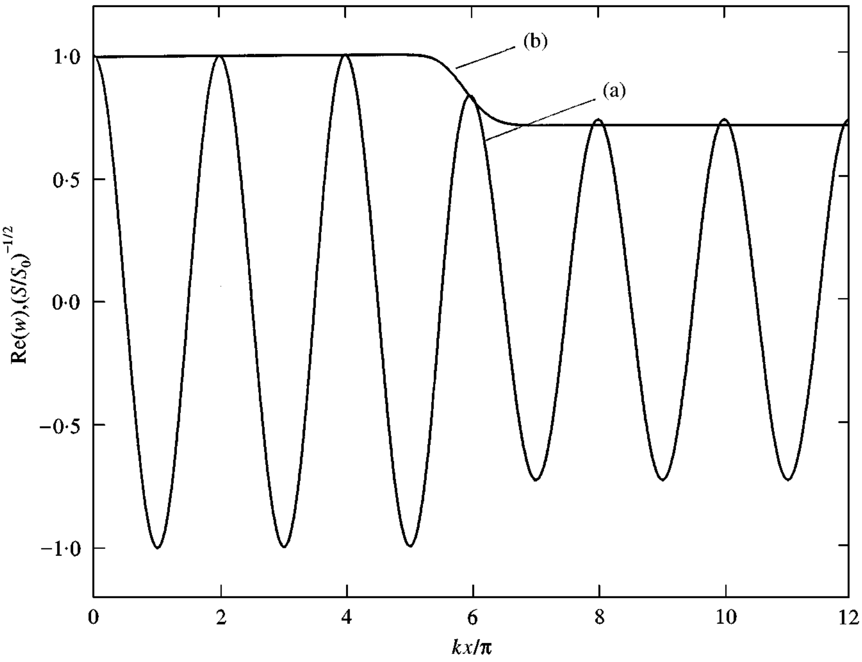



Figure 3. Wave propagation through a change of cross-sectional area in a rod, $n_\lambda = 1$ and $S_R = 2$. The plot shows: (a) the real part of the displacement, $\mathrm{Re}(w)$; (b) the square root of the area variation, $(S/S_0)^{-1/2}$.

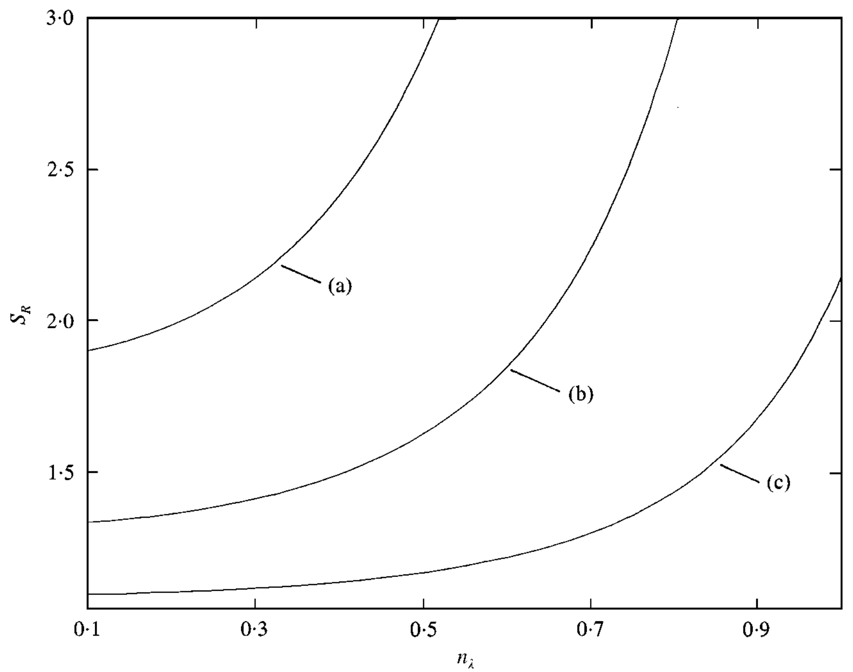



Figure 4. Contours of the incident wave amplitude $|b_1|$ required to produce a transmitted wave of unit amplitude in a rod, for various combinations of $n_\lambda$ and $S_R$. (a) $|b_1| = 1{\cdot}05$; (b) $|b_1| = 1{\cdot}01$; (c) $|b_1| = 1{\cdot}001$.

approximate theory. A contour plot of this quantity is shown in Figure 4 for a range of values of $n_\lambda$ and $S_R$. The selected range of $n_\lambda$ and $S_R$ represents a fairly severe test of the theory, since in most cases the waveguide undergoes a significant change in cross-section within the space of wavelength. Nonetheless, the error (i.e., the difference in $|b_1|$ from unity at $kx = 12\pi$) is very small over a significant portion of the diagram. As discuss in the following section, higher order terms in equations (34) and (35) can be used to estimate the wave reflection and transmission coefficients at a more rapid change in the waveguide cross-section.

### 3.3. REFLECTION AND TRANSMISSION OF A PROPAGATING WAVE

If the rate of change of the waveguide properties is such that significant energy transfer between the various wave components can occur, then a more detailed analysis of the system is required. The *exact* solution to equation (24) will have the form

$$\mathbf{b}(X) = \mathbf{W}(0, X)\mathbf{b}(0), \tag{45}$$

where $\mathbf{W}$ is the transfer matrix (expressed in terms of the wave amplitudes rather than the state vector $\mathbf{u}$) of that section of the waveguide which lies between $x = 0$ and $x = X$. It should be emphasized that $\mathbf{W}$ differs from the matrix $\mathbf{T}$ which appears in equation (8), and in particular $\mathbf{W}$ may not be symplectic or K-unitary when expressed in terms of wave amplitudes. In what follows, it will be assumed that the waveguide which lies outside the region $0 < x < X$ is homogeneous, and the aim is to investigate the wave scattering properties of the section $0 < x < X$. To this end, equation (45) can be partitioned in the form

$$\begin{pmatrix} \mathbf{b}_-(X) \\ \mathbf{b}_+(X) \end{pmatrix} = \begin{pmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{b}_-(0) \\ \mathbf{b}_+(0) \end{pmatrix}, \tag{46}$$

where $\mathbf{b}_-$ contains those wave components which either propagate energy to the left or decay to the left, and conversely $\mathbf{b}_+$ contains those wave components which either propagate energy to the right or decay to the right. If a left travelling wave is incident from the region $x > X$, then this wave will be partly reflected and partly transmitted; for $x < 0$ the resulting wave motion must either propagate energy to the left or decay to the left, and this gives the boundary condition $\mathbf{b}_+(0) = 0$. Equation (46) then yields

$$\mathbf{b}_-(0) = \mathbf{W}_{11}^{-1}\mathbf{b}_-(X), \quad \mathbf{b}_+(X) = \mathbf{W}_{21}\mathbf{W}_{11}^{-1}\mathbf{b}_-(X), \tag{47a, b}$$

where $\mathbf{b}_-(X)$ can be specified in terms of the incoming wave. Now for a slowly varying waveguide the transfer matrix which appears in equation (46) can be expanded in the form of a perturbation series so that

$$\mathbf{W} = \mathbf{W}_{(0)} + \varepsilon\mathbf{W}_{(1)} + \varepsilon^2\mathbf{W}_{(2)} + \cdots. \tag{48}$$

where $\mathbf{W}_{(n)}$ represent the $n$th order transfer matrix. It then follows directly from equations (34) and (35) that

$$\mathbf{W}_{(0)} = \boldsymbol{\Gamma}(x), \quad \mathbf{W}_{(n)} = \boldsymbol{\Gamma}(x)\int_0^x \boldsymbol{\Gamma}^{-1}(x')\mathbf{B}(x')\mathbf{W}_{(n-1)}(x')\,\mathrm{d}x', \quad n > 0, \quad (49, 50)$$

and, to second order, equation (47b) can be written in the form

$$\mathbf{b}_+(X) = \{\varepsilon\mathbf{W}_{(1)21} + \varepsilon^2[\mathbf{W}_{(2)21} - \mathbf{W}_{(1)21}\boldsymbol{\Gamma}_{11}^{-1}\mathbf{W}_{(1)11}]\}\boldsymbol{\Gamma}_{11}^{-1}\mathbf{b}_-(X). \quad (51)$$

Furthermore, it follows from equations (47a) and (48) that

$$\mathbf{b}_-(0) = \{\mathbf{I} - \varepsilon\boldsymbol{\Gamma}_{11}^{-1}\mathbf{W}_{(1)11} + \varepsilon^2(\boldsymbol{\Gamma}_{11}^{-1}\mathbf{W}_{(1)11})^2 - \varepsilon^2\boldsymbol{\Gamma}_{11}^{-1}\mathbf{W}_{(2)11}\}\boldsymbol{\Gamma}_{11}^{-1}\mathbf{b}_-(X). \quad (52)$$

Equations (49–52) are applied to a rod and a beam respectively in the following sections.

3.3.1. *Application to a rod*

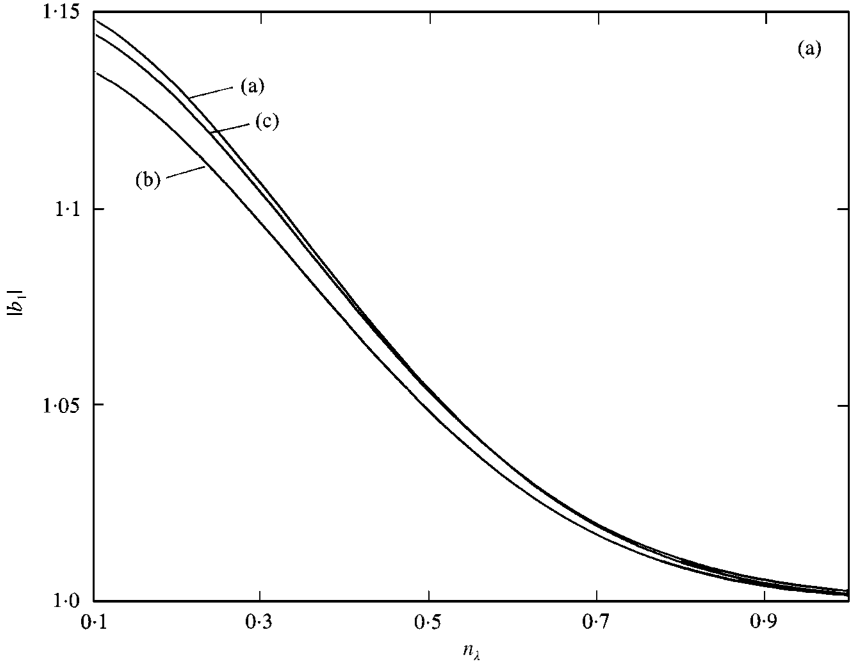
Equations (49–52) can be applied to a rod by noting that the entries of the matrices $\boldsymbol{\Gamma}$ and $\mathbf{B}$ can in this case be deduced from equations (41) and (43). The second order contribution to equation (51) is zero, and the first order component yields

$$b_2(X) = \beta(X)\exp(-2\mathrm{i}kX)b_1(X), \quad \beta(x) = \int_0^x (S'/2S)\exp(2\mathrm{i}kx')\,\mathrm{d}x', \quad (53, 54)$$

where $S' = \mathrm{d}S/\mathrm{d}x'$ and the notation of section 3.2 ($b_1 \equiv b_-$ and $b_2 \equiv b_+$) has been adopted. In contrast, the first order contribution to equation (52) is zero, and the terms of zero and second order give

$$b_1(0) = \left(1 - \int_0^X (S'/2S)\beta(x')\exp(-2\mathrm{i}kx')\,\mathrm{d}x'\right)\exp(-\mathrm{i}kX)b_1(X). \quad (55)$$

Equations (53) and (55) can each be used to obtain an improved estimate of the wave amplitude $b_1(X)$ for the case considered in section 3.2, i.e., a left travelling wave of unit amplitude at $x = 0$, so that $b_1(0) = 1$. The simplest estimate arises by considering equation (53) in conjunction with the energy flow condition $|b_1|^2 - |b_2|^2 = 1$, which yields $|b_1(X)| = 1 + |\beta|^2/2$. A more accurate second order approximation is furnished directly by the inverse of equation (55). Results yielded by these two estimates are shown in Figure 5 for the example system considered in section 3.2, with $S_r = 3$. It can be seen from Figure 5(a) that the magnitude of $b_1$ is well predicted by the second order approximation, while the result based on equation (53) is poorer but generally within 1·2% of the exact value. The phase of $b_1$ as predicted by equation (55) is compared with the exact result in Figure 5(b), and again good agreement is obtained.



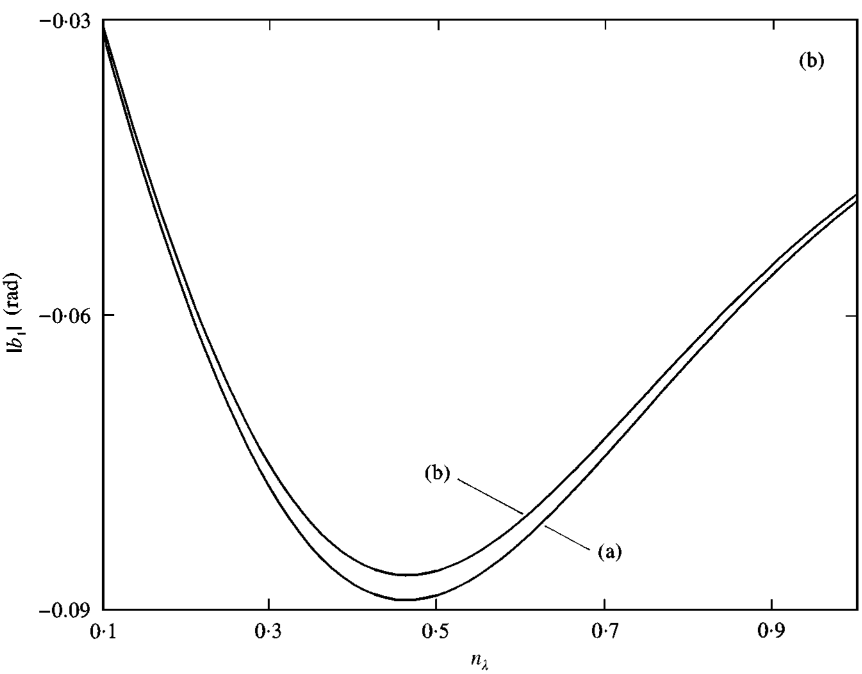




Figure 5. (a) Incident wave amplitude $|b_1|$ required to produce a transmitted wave of unit amplitude in a rod for the case $S_R = 3$. The three curves shown are: (a) the exact result; (b) the approximate result yielded by equation (53); (c) the approximate result yielded by equation (55). (b) Phase of the incident wave relative to the transmitted wave for the case considered in Figure 5(a). The two curves shown are: (a) the exact result; (b) the approximate result yielded by equation (55).

3.3.2. *Application to a beam*

The equations of motion for a pre-compressed beam which rests on an elastic foundation are given by equations (4–6). The four eigenvalues $\lambda_j$ of the matrix which appears in equation (6) yield the four wavenumbers $k_j$ of the beam in the form $k_j = \mathrm{i}\lambda_j$. In describing the behaviour of the beam, it is convenient to introduce the two non-dimensional parameters $\gamma_K = K/\rho S\omega^2$ and $\gamma_C = C/\rho S\omega^2$. It can be shown that (i) for $\gamma_K < 1$ two of the wavenumbers are real, (ii) for $1 < \gamma_K < 1 + \gamma_C^2/4$ all four wavenumbers are real, and (iii) for $\gamma_K > 1 + \gamma_C^2/4$ none of the wavenumbers are real. This behaviour is shown in Figure 6 for the particular case $\gamma_C = 2$. The present section is concerned with wave propagation in a non-uniform beam with $\gamma_C = 2$ and $\gamma_K < 1$ so that there are always two propagating waves and two evanescent waves: the special case of transition through the cut-off point $\gamma_K > 1 + \gamma_C^2/4 = 2$ is considered in section 4.

In what follows, the stiffness of the elastic foundation is taken to vary with position $x$ so that the non-dimensional stiffness $\gamma_K(x)$ varies in accordance with equation (44), albeit with $S$ replaced by $\gamma_K$ and $S_0$ and $S_1$ replaced by $\gamma_{K0}$ and $\gamma_{K1}$ respectively. As in section 3.2, the parameter $n_\lambda = 3k\sigma/\pi$ is used to give an indication of the number of wavelengths taken to effect the change in $\gamma_K$, although in the present case the wavenumber $k$ varies with $x$—the value of $k$ at $\gamma_{K0}$ is taken as the reference in defining $n_\lambda$. Three cases are considered here, these being $(\gamma_{K0}, \gamma_{K1}, n_\lambda) = (0{\cdot}9, 0{\cdot}0, 0{\cdot}2)$, $(0{\cdot}9, -50, 0{\cdot}2)$ and $(0{\cdot}9, -50, 1{\cdot}0)$, where it can be noted that a negative value of $\gamma_K$ corresponds to mass loading rather than a spring support. The boundary conditions are taken to be $\mathbf{b}_+(0) = 0$ together with a left travelling incident wave of unit amplitude at $x = X$, so that $\mathbf{b}_-(X)$ is specified; the remaining wave components are then given by equations (47–52). The amplitude of the right travelling wave at $x = X$ (i.e., the reflected wave) is given in Table 1 for each of the three cases considered. The exact result is shown (obtained by direct numerical integration of the governing equations), together with successive approximations obtained by including terms of up to eighth order in equations (47b) and (48). Also shown in Table 1 are the corresponding results for the amplitude of the transmitted wave, obtained in this case from equations (47a) and (48). The displacement of the system is shown in Figure 7 for the third example case—it is clear that although the transmission coefficient is close to unity when expressed in terms of the wave amplitudes, the variation in the beam properties has a marked effect on the physical deflection.

One obvious feature of the results shown in Table 1 is that, unlike the rod example, the transmitted and reflected waves are in most cases poorly predicted by a second order approximation. This can be traced to the effect of the evanescent waves on the structure of the transfer matrix $\mathbf{W}$. This is shown in Figures 8 and 9 for the case $(\gamma_{K0}, \gamma_{K1}, n_\lambda) = (0{\cdot}9, -50, 1{\cdot}0)$. The results shown relate to the pertinent partitions of the matrix $\mathbf{\Gamma}^{-1}\mathbf{W}$; for a uniform beam the $2 \times 2$ partition $(\mathbf{\Gamma}^{-1}\mathbf{W})_{11}$ is equal to the identity matrix whereas the $2 \times 2$ partition $(\mathbf{\Gamma}^{-1}\mathbf{W})_{21}$ is equal to zero. Clearly, the matrix entries which relate to the evanescent waves differ considerably from the uniform beam case, and the series represented by equation (48) must be carried to sixth or even eighth order before these terms are accurately captured. Having said this, it can be noted that the case considered relates to a fairly severe

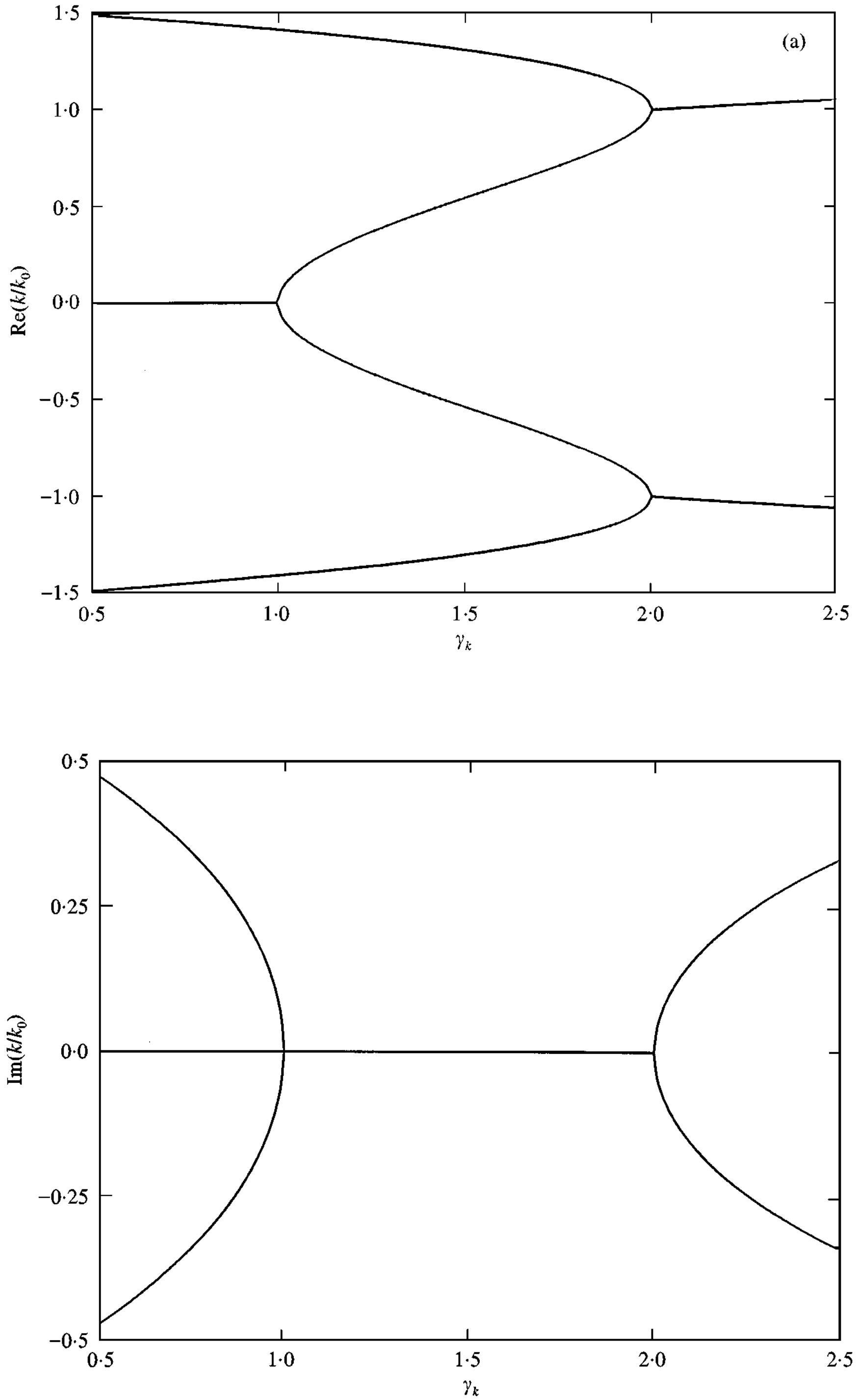



Figure 6. (a) The imaginary part of the eigenvalues (i.e. the real part of the wavenumbers) for a beam on an elastic foundation with $\gamma_C = 2$. The symbol $k_0$ represents the wavenumber for a simple beam, $EIk_0^4 = \rho S\omega^2$. (b) The real part of the eigenvalues (i.e., the imaginary part of the wavenumbers) for a beam on an elastic foundation with $\gamma_C = 2$.

TABLE 1

*Reflection and transmission coefficients for a beam on elastic foundations; the three cases are* $(\gamma_{K0}, \gamma_{K1}, n_\lambda) = (0{\cdot}9, 0{\cdot}0, 0{\cdot}2)$, $(0{\cdot}9, -50, 0{\cdot}2)$, *and* $(0{\cdot}9, -50, 1{\cdot}0)$, *and n represents the order of the perturbation expansion*

| | $\|r\|$ | | | $1/\|t\|$ | | |
|---|---|---|---|---|---|---|
| $n$ | Case 1 | Case 2 | Case 3 | Case 1 | Case 2 | Case 3 |
| 1 | 0·0418 | 1·0522 | 0·2341 | 0·9888 | 0·0544 | 0·0813 |
| 2 | 0·0347 | 0·0745 | 0·0121 | 1·00217 | 1·2612 | 1·0759 |
| 3 | 0·0374 | 0·3931 | 0·0813 | 1·00044 | 0·9457 | 0·9957 |
| 4 | 0·0372 | 0·2283 | 0·0549 | 1·00074 | 1·0900 | 1·0111 |
| 5 | 0·0373 | 0·2929 | 0·0636 | 1·00071 | 1·0335 | 1·0019 |
| 6 | 0·0373 | 0·2781 | 0·0623 | 1·00071 | 1·0472 | 1·0027 |
| 7 | 0·0373 | 0·2835 | 0·0627 | 1·00071 | 1·0425 | 1·0020 |
| 8 | 0·0373 | 0·2827 | 0·0627 | 1·00071 | 1·0432 | 1·0021 |
| Exact | 0·0373 | 0·2831 | 0·0627 | 1·00070 | 1·0427 | 1·0019 |

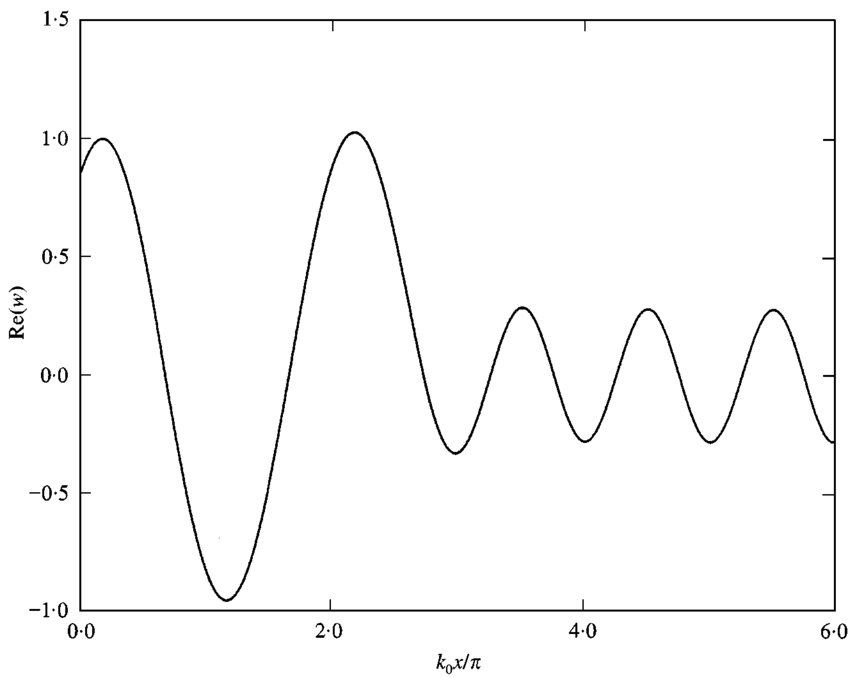



Figure 7. Wave propagation through a change in properties for a beam on an elastic foundation with $\gamma_C = 2$ and $(\gamma_{K0}, \gamma_{K1}, n_\lambda) = (0{\cdot}9, -50, 1{\cdot}0)$.

change in the beam properties, as can be seen from Figure 7. The change is less severe for the first example shown in Table 1, and in this case a second order approximation yields a good estimated of the reflected wave amplitude.

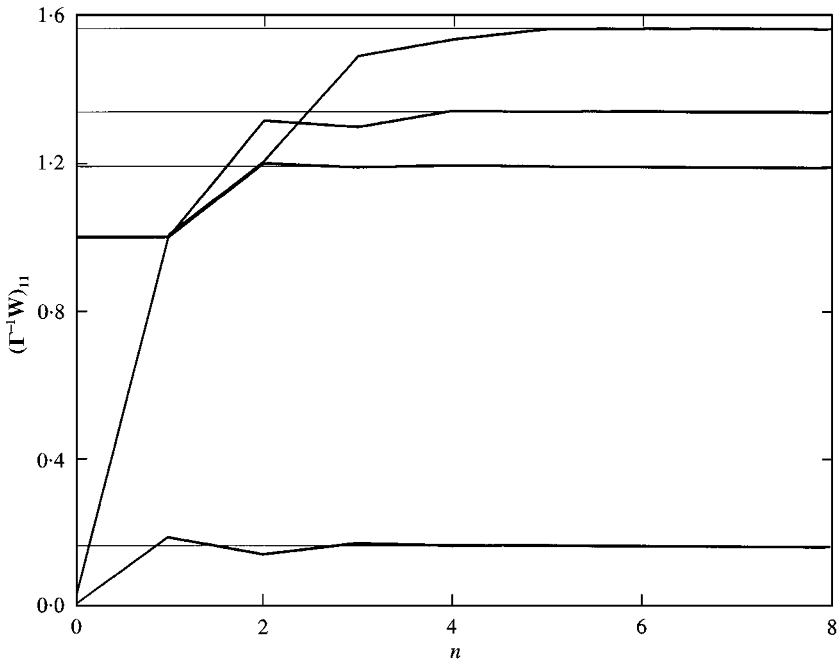



Figure 8. The four entries of the matrix partition $(\boldsymbol{\Gamma}^{-1}\mathbf{W})_{11}$ plotted as a function of the perturbation order $n$, defined such that terms of order $\varepsilon^n$ are retained in the analysis. The horizontal lines indicate the exact values of the matrix partition. $\gamma_C = 2$ and $(\gamma_{K0}, \gamma_{K1}, n_\lambda) = (9{\cdot}0,\ -50, 1{\cdot}0)$.

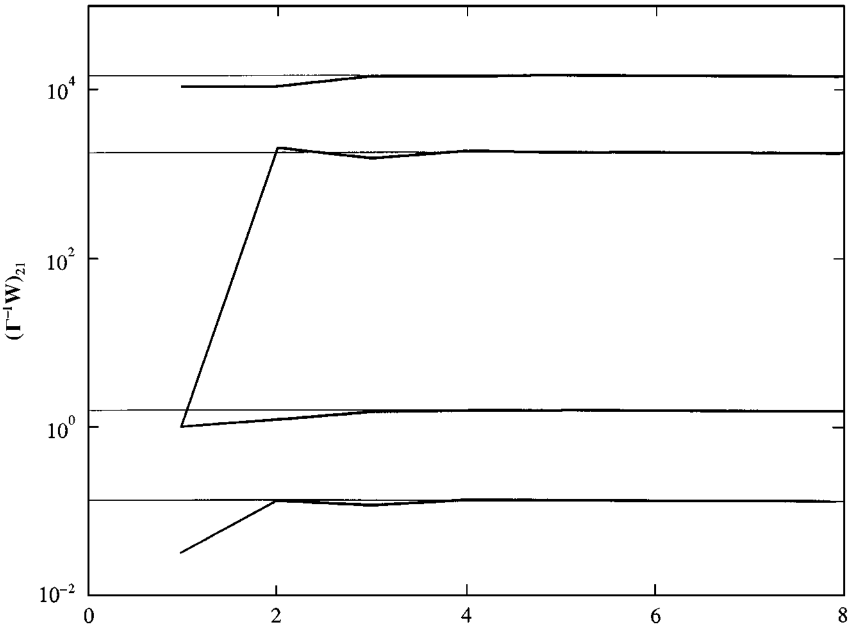



Figure 9. The four entries of the matrix partition $(\boldsymbol{\Gamma}^{-1}\mathbf{W})_{21}$ plotted as a function of the perturbation order $n$, defined such that terms of order $\varepsilon^n$ are retained in the analysis. The horizontal lines indicate the exact values of the matrix partition. $\gamma_C = 2$ and $(\gamma_{K0}, \gamma_{K1}, n_\lambda) = (9{\cdot}0,\ -50, 1{\cdot}0)$.

### 3.4. WAVE PROPAGATION IN A PERIODIC SYSTEM

The foregoing approximate theory can be applied to a system which has a periodic variation in properties, i.e., a system which can be considered to be composed of a number of identical units which are connected end to end. In this case, the wave bearing characteristics of the system are described by the eigenvalues of the transfer matrix of a single unit. For the specific example of a rod with a periodic variation in cross-sectional area, the eigenvalues ($\mu$ say) of the transfer matrix satisfy the equation [20]

$$\mu^2 - (1/T_u + 1/T_u^*)\mu + 1 = 0, \tag{56}$$

where $T_u$ is the wave transmission coefficient of the unit—if the $2 \times 2$ transfer matrix $\mathbf{W}$ is expressed in terms of wave amplitude coordinates, then $T_u = 1/W_{11}$. A second order approximation to $T_u$ can be deduced from equation (55) in the form

$$T_u = \left(1 - \int_0^X (S'/2S)\beta(x')\exp(-2\mathrm{i}kx')\,\mathrm{d}x'\right)\exp(-\mathrm{i}kX), \tag{57}$$

where $\beta(x)$ is given by equation (54). An alternative approximation, based on the principle of energy flow conservation combined with a first order estimate of the wave reflection coefficient, is given by

$$T_u = \exp(-\mathrm{i}kX)\sqrt{1 - |\beta(X)|^2}. \tag{58}$$

In this the result, the magnitude of $T_u$ is deduced from the magnitude of the reflection coefficient (given by $|\beta|$), while the phase is taken to be that of a uniform waveguide. To consider the specific example of a sinusoidal variation in cross-sectional area, so that $S = S_0(1 + \alpha_0 \sin k_0 x)$, it follows from equation (54) that

$$\beta(x) = \int_0^X \left\{\frac{\alpha_0 k_0 \cos(k_0 x')}{2[1 + \alpha_0 \sin(k_0 x')]}\right\}\exp(2\mathrm{i}kx')\,\mathrm{d}x', \tag{59}$$

with $X = 2\pi/k_0$. Equation (56) yields an eigenvalue pair with the structure ($\mu$, $1/\mu$), where $\mu$ is conventionally expressed in terms of an attenuation constant $\delta_a$ and a phase constant $\varepsilon_p$ in the form $\mu = \exp(\mathrm{i}\varepsilon_p + \delta_a)$. For a uniform waveguide with $\alpha_0 = 0$, the eigenvalues are just $\mu = \exp(\pm\mathrm{i}2\pi k/k_0)$ so that $\delta_a = 0$ and $\varepsilon_p$ increases linearly with $k$—when phase wrapped over the range 0 to $\pi$, the plot of $\varepsilon_p$ against $k$ has the appearance of a saw-tooth curve. In contrast, exact and approximate results for the attenuation and phase constants are shown in Figure 10 for the non-uniform case $\alpha_0 = 0{\cdot}4$. It can be seen that the results based on equation (57) represent a good estimate of the phase constant and a somewhat less accurate estimate of the attenuation constant. The results based on the lower order approximation, equations (58) and (59), also provide a good estimate of the behaviour of the phase and attenuation constants. By performing a Taylor series expansion of the denominator of equation (59), it is clear that the structural

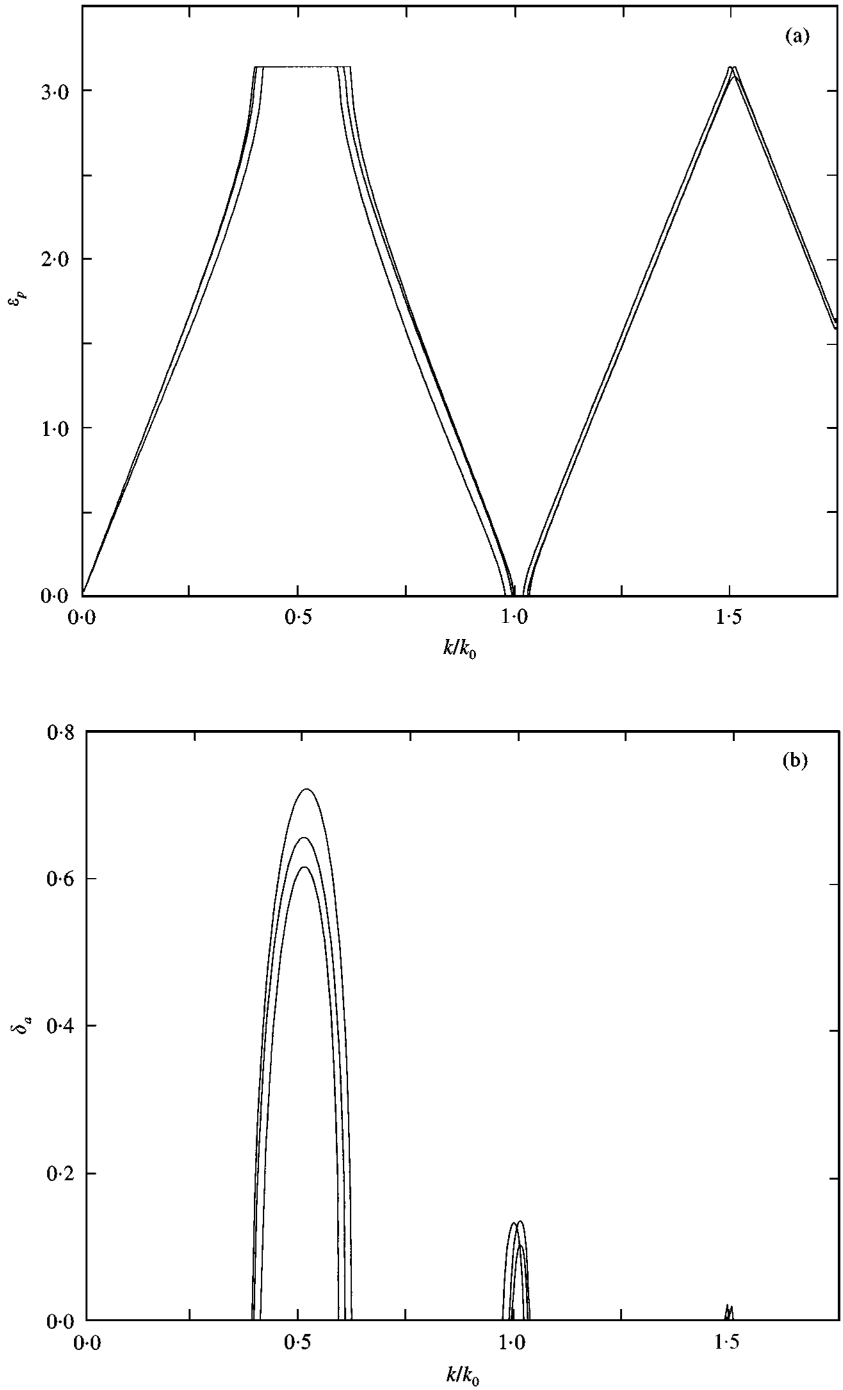



Figure 10. (a) The phase constant $\varepsilon_p$ for a periodic rod. The three curves shown correspond to: (a) the exact result; (b) the result yielded by equation (57); (c) the result yielded by equation (58). To distinguish the curves, for $0 \leqslant k/k_0 \leqslant 1$, curve (c) is uppermost, curve (a) is the middle curve, and curve (b) is the lower curve. For $1 \leqslant k/k_0 \leqslant 1{\cdot}5$, curve (b) is uppermost. (b) The attenuation constant $\delta_a$ for a periodic rod. The three curves shown correspond to (a) the exact result; (b) the result yielded by equation (57); (c) the result yielded by equation (58). The curves are ordered as per Figure 10(a).

periodicity will have most effect for frequencies with $k = nk_0/2$, where $n$ is an integer. It is also clear that the magnitude of the effect will decrease with increasing $n$, and this behaviour is visible in Figure 10. Thus, a low order approximate analysis of the present type can readily predict the key features of the periodic system behaviour.

### 3.5. WAVE PROPAGATION IN A RANDOM SYSTEM

There has been much previous interest in the statistics of wave propagation in random media, with application to both solid-state physics (see e.g., reference [13]) and applied mechanics (see, e.g., reference [22]). Scott [11] has considered the propagation of waves in a random waveguide with the governing equation

$$\mathrm{d}^2 w/\mathrm{d}x^2 + [k^2 + N(x)]w = 0, \tag{60}$$

where $N(x)$ is a stationary Gaussian random process with zero mean. Equation (60) has the form of equation (2), and it can be readily be cast into the form of equation (3) to allow application of the present theory. It follows that in this case

$$\beta(x) = (4k^2)^{-1} \int_0^x (\partial N/\partial x')\exp(2\mathrm{i}kx')\,\mathrm{d}x' \tag{61}$$

For small $\beta$ equation (58) is valid, and it follows that the transmission coefficient of a length $L$ of the waveguide has the property

$$E[|T_u|^2] = 1 - E[|\beta(L)|^2], \tag{62}$$

where $E[\ ]$ represents the expected value. Equation (61) implies that

$$E[|\beta(L)|^2] = (4k^2)^{-2} \int_0^L \int_0^L (\partial N/\partial x_1)(\partial N/\partial x_2)\exp[2\mathrm{i}k(x_1 - x_2)]\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{63}$$

and if $L$ is much greater than the correlation length of $N$ then it follows that

$$E[|T_u|^2] = 1 - (4k^2)^{-1} S_{NN}(2k)L, \tag{64}$$

where $S_{NN}(k)$ is the spectral density of $N(x)$. It can be noted that equation (64) can also be derived from the second order approximation to $T_u$, equation (57).

Equation (64) will be valid only if the condition $(4k^2)^{-1}S_{NN}(2k)L \ll 1$ is met, since the foregoing derivation is based on a low order perturbation expansion of the system transfer matrix. For a very long waveguide this condition will *not* hold, even though the waveguide is considered to be weakly disordered so that $(4k^2)^{-1}S_{NN}(2k)$ is a small parameter. For a long system composed of $M$ statistically independent sections of length $L$, it can be shown that the transmission coefficient $T_M$ has the property (e.g. equation (8) of reference [22])

$$\ln|T_M|^2 = M\ln|T_u|^2 + O(M^{1/2}), \tag{65}$$

and thus

$$E[\ln|T_M|^2] \approx -(4k^2)^{-1} S_{NN}(2k)ML. \tag{66}$$

It is known that for large $M$ the statistical distribution of $\ln|T_M|^2$ becomes Gaussian, and furthermore one-parameter scaling applies, so that the variance is equal to twice the mean value [23]. Equation (66), in conjunction with a one-parameter Gaussian distribution for $\ln|T_M|^2$, is in full agreement with equation (3.27) of reference [11]. It is interesting to note that the reduction in transmission due to irregularity is dominated by the spectral component of $N(x)$ which has twice the homogeneous wavenumber $k$; this is consistent with the analysis of section 3.4, where it was shown that periodicity with wavenumber $k_0$ produces greatest effect when $k = k_0/2$.

For more general random waveguides, equation (52) may be used in place of equation (57) or (58) to develop the transmission properties of a section of length $L$, where $L$ is short enough to ensure that the perturbation series is accurate. If $L$ exceeds the correlation length of the random variations, then the transmission properties of a section of length $ML$ can be analyzed by considering $M$ sequential uncorrelated sections of length $L$. This analysis will lead to the consideration of the statistics of the product of $M$ uncorrelated transfer matrices, and this problem is addressed in reference [24, 25] for example.

## 4. WAVE REFLECTION AT A CUT-OFF CROSS-SECTION

The foregoing analysis has considered wave propagation, transmission, and reflection in an inhomogeneous waveguide; thus far it has been assumed that waves can propagate along all parts of the waveguide, and the aim has been to estimate the spatial evolution of the wave amplitudes. In some situations, however, a change in the waveguide properties can lead to a fundamental change in the nature of the wave motion, and in this case the previous analysis is not applicable. One example of this type of behaviour is shown in Figure 6 for the case of a beam on an elastic foundation. For $\gamma_K < 2$ there are four propagating waves, whereas for $\gamma_K > 2$ there are four evanescent waves. The point $\gamma_K = 2$ represents a cut-off cross-section, in the sense that wave propagation to the right of this point cannot occur. Clearly, an incident right propagating wave will be fully reflected at $\gamma_K = 2$, and the phase of the reflected wave is of interest. In this section, a general analysis of wave reflection at a cut-off cross-section is presented, and it will be assumed without loss of generality that the $x$ co-ordinate is aligned such that $x = 0$ at the cut-off point.

When expressed in terms of wave amplitudes, the equations of motion of an inhomogeneous waveguide are given by equation (24), and the off-diagonal coefficients which appear in this equation are given by equation (26). Equation (26) can also be written in the form

$$[\mathbf{\Phi}^{-1}(\mathrm{d}\mathbf{\Phi}/\mathrm{d}x)]_{jr} = Q_{jr}/(\lambda_r - \lambda_j), \quad Q_{jr} = [\mathbf{u}_{j'}^{*\mathrm{T}}\mathbf{J}(\mathrm{d}\mathbf{A}/\mathrm{d}x)\mathbf{u}_r]/(\mathbf{u}_{j'}^{*\mathrm{T}}\mathbf{J}\mathbf{u}_j), \quad j \neq r, \tag{67, 68}$$

where, as defined in section 2.5, $\mathbf{u}_r$ is the right eigenvector associated with the eigenvalue $\lambda_r$, and $\mathbf{u}_{j'}$ is the right eigenvector associated with the eigenvalue $-\lambda_j^*$ (for a propagating wave $-\lambda_j^* = \lambda_j$, so that $j' = j$). Now a cut-off cross-section is characterized by the fact that two eigenvalues associated with propagating waves coalesce to produce a repeated eigenvalue, as can be seen in Figure 6. It follows from equation (67) that the coupling terms between two such waves will become very large near to the cut-off point (where $\lambda_r \approx \lambda_j$); if the scaling represented by equation (28) is adopted then the diagonal terms on the right sides of the equations of motion will remain small, and the coupling terms will become dominant as $\lambda_r \to \lambda_j$.

The detailed behaviour of the coupling terms in the vicinity of the cut-off point can be determined by noting initially that $Q_{jr} \to Q_{jj}$ as $\lambda_r \to \lambda_j$. Now by putting $\lambda_j = -\mathrm{i}k_j$, where $k_j$ is the (real) wavenumber associated with wave component $j$, it is possible to write $\mathrm{d}\mathbf{A}/\mathrm{d}x = (\partial\mathbf{A}/\partial\omega)c_{gj}(\partial k_j/\partial x)$, where $c_{gj} = (\partial\omega/\partial k_j)$ is the group velocity. It then follows from equation (68) that

$$Q_{jr} \to Q_{jj} = \mathbf{u}_j^{*\mathrm{T}}\mathbf{J}(\partial\mathbf{A}/\partial\omega)\mathbf{u}_j c_{gj}(\partial k_j/\partial x)/(\mathbf{u}_j^{*\mathrm{T}}\mathbf{J}\mathbf{u}_j). \tag{69}$$

Now equation (17) implies that the first group of terms in this result can be expressed as $\mathbf{u}_j^{*\mathrm{T}}\mathbf{J}(\partial\mathbf{A}/\partial\omega)\mathbf{u}_j = -(8/\omega)\bar{T}$, where $\bar{T}$ is the kinetic energy density. Furthermore, the quantity $2\bar{T}c_{gj}$ is the wave energy flow $P_{jj}$, which is given by equation (22). The net result is that equation (69) can be expressed in the very simple form

$$Q_{jr} \to \mathrm{i}(\partial k_j/\partial x). \tag{70}$$

In the immediate vicinity of the cut-off point ($x = 0$) the two wavenumbers $k_j$ and $k_r$ will have the form $k_j(x) = k_0 - f(x)$ and $k_r(x) = k_0 + f(x)$, where $k_0$ is the repeated wavenumber at cut-off and $f(x) \to 0$ with $\mathrm{d}f(x)/\mathrm{d}x \to \infty$ as $x \to 0$. The symmetric structure of the two wavenumbers around the limiting value $k_0$ is a direct consequence of the fact that the function $x(k)$ has a local maximum at $k = k_0$, as shown in Figure 6 (where the point $k = k_0$ occurs at $\gamma_K = 2$). It now follows from equations (24), (67) and (69) that in the vicinity to the left of the cut-off point ($x < 0$) the dominant terms in the equations of motion take the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} b_j \\ b_r \end{pmatrix} = \begin{pmatrix} 0 & f'(x)/2f(x) \\ f'(x)/2f(x) & 0 \end{pmatrix}\begin{pmatrix} b_j \\ b_r \end{pmatrix}, \tag{71}$$

where $f'(x) = \mathrm{d}f/\mathrm{d}x$. The general solution to this equation is

$$b_{Lj} = a_1\sqrt{f(x)} + a_2/\sqrt{f(x)}, \quad b_{Lr} = a_1\sqrt{f(x)} - a_2/\sqrt{f(x)}, \tag{72, 73}$$

where $a_1$ and $a_2$ are constants of integration, and the subscript $L$ indicates that the solution applies in the region to the left of the cut-off point. Clearly, the wave amplitudes are singular at the cut-off point, where $f(x) = 0$. This does not however imply that the physical response of the waveguide is singular: it can be deduced that the eigenvectors (when scaled in accordance with equation (28)) have the form

$$\mathbf{u}_{Lj} \approx [\mathbf{u}_0 + \mathrm{i}\mathbf{u}'_0 f(x)]/\sqrt{f(x)}, \quad \mathbf{u}_{Lr} \approx [\mathbf{u}_0 - \mathrm{i}\mathbf{u}'_0 f(x)]/\sqrt{f(x)}, \qquad (74, 75)$$

where $\mathbf{u}_0$ is the eigenvector associated with the cut-off point $k = k_0$, and $\mathbf{u}'_0$ is the derivative of this vector with respect to the eigenvalue $\lambda = -\mathrm{i}k$. Equations (72–75) imply that the physical response of the waveguide, $b_j\mathbf{u}_j + b_r\mathbf{u}_r$, remains finite at the cut-off point.

To the right of the cut-off point ($x > 0$) the two wave components $j$ and $r$ become evanescent. In this case, the wavenumbers can be written as $k_j(x) = k_0 - \mathrm{i}f(x)$ and $k_r(x) = k_0 + \mathrm{i}f(x)$, and equations (72)–(75) are replaced by

$$b_{Rj} = c_1\sqrt{f(x)} + c_2/\sqrt{f(x)}, \quad b_{Rr} = c_1\sqrt{f(x)} - c_2/\sqrt{f(x)}, \qquad (76, 77)$$

$$\mathbf{u}_{Rj} \approx [\mathbf{u}_0 - \mathbf{u}'_0 f(x)]/\sqrt{f(x)}, \quad \mathbf{u}_{Rr} \approx [\mathbf{u}_0 + \mathbf{u}'_0 f(x)]/\sqrt{f(x)}, \qquad (78, 79)$$

where $c_1$ and $c_2$ are constants. For the physical displacement $b_j\mathbf{u}_j + b_r\mathbf{u}_r$ to be continuous across the cut-off point, it follows from equations (72–79) that $a_1 = c_1$ and $a_2 = \mathrm{i}c_2$, where it has been noted that $\mathbf{u}_0$ and $\mathbf{u}'_0$ are independent vectors. Now equations (72–79) are valid in the immediate vicinity of the cut-off point, $0 \leqslant f(x) \leqslant f_0$ say, in which the wavenumbers change rapidly. To the right of the cut-off point, the amplitude of the evanescent term $b_{Rr}$ should become zero to ensure that displacement of the system remains finite with increasing $x$. If $b_{Rr}$ is taken to be near zero at $f(x) = f_0$, then it follows from equation (77) that $c_2/c_1 \approx f_0$. Equations (72) and (73) then yield $b_{Lr} = -\mathrm{i}b_{Lj}$ at $f(x) = f_0$, from which it can be deduced that the cut-off reflection coefficient is $-\mathrm{i}$, so that the reflected wave ($b_{Lr}$) is phase-shifted by $-\pi/2$ relative to the incident wave ($b_{Lj}$).

An example of wave reflection at a cut-off cross-section is shown in Figure 11. The case considered is a beam on an elastic foundation with the properties described in section 3.3.2. The foundation stiffness parameter $\gamma_K$ is varied over the range $1{\cdot}5 \leqslant \gamma_K \leqslant 2{\cdot}5$, which covers the cut-off point $\gamma_K = 2$. The $x$-coordinate shown on the figure is such that cut-off occurs at $x = 10$, and the change in the foundation stiffness occurs predominantly over the region $7{\cdot}5 \leqslant x \leqslant 12{\cdot}5$. It can be seen from Figure 6 that the beam wavenumbers change gradually away from the immediate vicinity of the cut-off point, where a very rapid rate of change occurs; this is reflected in the behaviour of the propagating wave components which are shown in Figure 11. Three significant points which illustrate the foregoing theory can be noted from Figure 11: (i) the incident and reflected waves have different wavenumbers $k_j$ and $k_r$, and hence different wavelengths; (ii) the wave amplitudes become infinite at the cut-off-point, but combine to give a finite physical displacement; (iii) the phase of the reflection is $-\pi/2$. As a point of detail, Figure 11

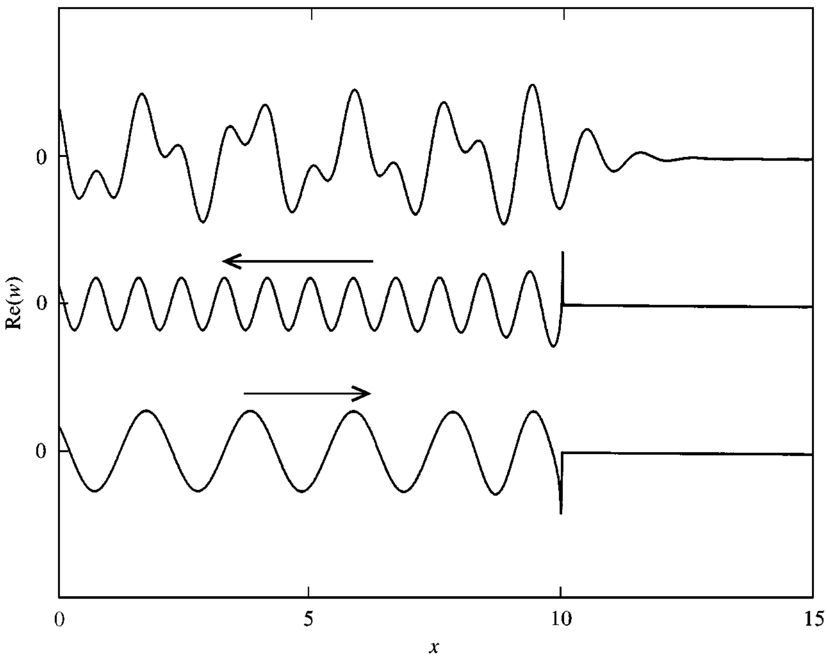



Figure 11. Wave reflection at a cut-off cross-section in a beam. Upper curve—the real part of the beam displacement; middle curve—the real part of the displacement arising from the reflected wave; lower curve—the real part of the displacement arising from the incident wave. All three curves are shown to the same vertical scale.

was constructed by integrating the beam equations of motion, equations (4–6), backwards in space, under the initial condition of a decaying evanescent wave at $x = 15$. The propagating wave components were then extracted from the complete response to the left of the cut-off point.

A further example of wave reflection at a cut-off cross-section has been given by Scott and Woodhouse [14]. The case considered was a plate strip of varying curvature, which behaves in a similar way to the musical saw. It was shown that the reflection coefficient at the cut-off cross section is indeed $-\mathrm{i}$, and this fact was used in conjunction with the principle of phase closure to estimate the higher natural frequencies of the structure.

## 5. CONCLUDING REMARKS

A method has been presented for the analysis of wave motion in an inhomogeneous waveguide. The method is based on a first order representation of the waveguide equations of motion in the form of equation (1). It has been shown that the matrix **A** which appears in this equation satisfies equation (12) for a symmetric system and equation (15) for a conservative system—these results are the counterpart of the transfer matrix of a finite section of the waveguide being symplectic and K-unitary respectively. The kinetic energy density of a conservative system is given by equation (17), and wave energy flow has been discussed in section 2.5 in the context of the properties of the eigenvalues and eigenvectors of the matrix **A**.

The equations of motion are conveniently expressed in terms of wave amplitudes via equation (24), and a perturbation solution of this form of the equations has been presented in section 3.1. It has been shown that the approach may be applied to deterministic, periodic, and random waveguides, and in most cases a first or second order solution reveals the key features of the physical nature of the system behaviour. Finally, the issue of wave reflection at a cut-off cross-section has been addressed, and it has been shown that the phase of the reflection coefficient is $-\pi/2$.

The method presented here can be applied to any inhomogeneous waveguide which is governed by equation (1); this covers a much broader class of system than the examples considered here, and future work could employ the method to consider localization and cut-off phenomena in various other structural and acoustic systems. The present work has been concerned primarily with conservative systems, although damping could readily be introduced into the matrix **A** which appears in equation (1). The eigenvectors which appear in section 2.4 will then represent damped wave motion, and some aspects of the energy flow for this case have been discussed in reference [17]. For light damping the main effect will be to modify the eigenvalues which appear in the matrix $\mathbf{\Lambda}$ in equation (24), and this can readily be incorporated in the perturbation scheme presented in section 3.1.

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